An Integrated Hub-Location, Ship-Routing and Berth-Allocation Model for a Maritime Network

Mathematical Model

1 Overview

This document presents the mathematical formulation of an integrated maritime logistics optimization model. The model jointly addresses three decisions that are usually studied in isolation:

- **Hub location and allocation** — choosing which ports become hubs and assigning every demand node to exactly one hub (a single-allocation p-hub design);
- **Ship routing and chartering** — deciding how many vessels to charter on each inter-hub link and routing flow through the hub network; and
- **Berth allocation and scheduling** — assigning each arriving and departing vessel to a section of a port's quay and sequencing the loading and delivery operations in time so that vessels sharing a berth section do not overlap.

Cargo moves in three legs: trucks carry it from origin (non-hub) nodes to a hub, chartered vessels carry it between hubs, and trucks carry it from the destination hub to the destination node. The objective minimizes the total system cost—first-mile trucking, last-mile trucking, and the inter-hub vessel cost (transfer, charter, loading, and delivery)—while a rich set of timing constraints synchronizes truck arrivals, vessel availability, berthing, processing, and final delivery against customer order deadlines.

2 Problem Description

The network consists of a set of nodes, some of which are selected as hub ports. Non-hub nodes are connected to hubs by truck; hubs are connected to one another by chartered vessels. Each node both ships cargo (to be picked up) and receives cargo (to be delivered). Two flavors of vessel appear at any hub i: *dispatched* vessels (those departing i toward another hub j) and *received* vessels (those that have arrived at i from another hub k). Both must be berthed at i, which couples the routing decisions to the berth-allocation decisions.

At each port the quay is divided into discrete **berth sections**. A vessel occupies a contiguous block of sections for the duration of its loading or delivery operation, and two vessels whose section blocks overlap may not be processed at the same time. The model therefore contains disjunctive (big-M) sequencing constraints that, for every pair of vessels sharing a section, force one to be fully processed before the other begins. Four such relations are needed, one for each ordered combination of dispatched and received vessels.

2.1 Modeling Assumptions

The following assumptions are embedded in the formulation:

1. Exactly P hubs are opened; each non-hub node is assigned to a single hub (single allocation).
2. Inter-hub transport uses chartered, capacitated vessels; the number chartered on any link may not exceed a maximum MNV.
3. Routing through the network is single-hub on the relevant legs (each shipment passes through its assigned hub).
4. Each vessel begins its loading and its delivery operation at exactly one time point of the planning horizon.
5. A vessel's berthing duration equals its processing (loading or delivery) duration.
6. The total berth sections occupied at a port cannot exceed the quay length.
7. Cargo must reach each customer no later than that customer's order deadline.

3 Notation

Symbol	**Definition**
Sets and indices	
N	Set of (non-hub) nodes; indices m, n
H	Set of candidate hub ports; indices i, j, k
T	Set of time periods in the planning horizon; index t
B_i	Set of berth sections at port i; index b
NV_{ij}	Set of vessels operating on inter-hub link $i \to j$; indices v, v'
Cost and capacity parameters	
C_{mi}, C_{nj}	Truck travel cost node→hub ($m \to i$) and hub→node ($j \to n$)
C_{ij}	Inter-hub transfer cost from hub i to hub j
RV	Vessel charter (rental) cost
PP_{ijv}, PD_{ijv}	Cost of the loading process at i and the delivery process at j
NTP_m, NTD_n	Number of trucks performing pickup/loading and delivery
CB	Vessel capacity
MNV	Maximum vessels chartered on a single inter-hub link
P	Number of hubs to be opened
L_i	Quay length (number of berth sections) at port i
M	Large positive constant (big-M)
O_m, W_{mn}	Supply originating at node m; demand from m to n (flow balance)
Time parameters	
td_{ijv}	Sailing time from port i to port j
tt_{mi}, tt_{nj}	Truck travel time node→hub and hub→node
Dt_{mi}	Truck unloading time at port i
Pt_{nj}	Truck loading duration
PP_{ijv}, PD_{ijv}	Loading and delivery process durations of vessel v
At_{mn}	Order deadline for moving cargo from node m to node n
n_{bv}	Number of berth sections occupied by vessel v starting at section b

Symbol	Definition
Time variables	
A_{ijv}	Arrival time of vessel v (route $i \to j$)
A_{iiv}	Availability time of vessel v at its origin port i
tSP_{ijtv}	Start time of the loading process of vessel v at port i
tSD_{ijtv}	Start time of the delivery process of vessel v
D_{ijv}	Completion time of delivery at destination port j
CD_{mijn}	Time at which cargo reaches customer n
Decision variables	
x_{ii}	$= 1$ if node i is opened as a hub
x_{mi}	$= 1$ if node m is assigned to hub i
NV_{ij}	Integer; number of vessels chartered on link $i \to j$
f_{ijm}	Continuous flow of node-m cargo routed through hubs i and j
SP_{ijtv}	$= 1$ if vessel v (route $i \to j$) begins loading at time t
SD_{kitv}	$= 1$ if vessel v (route $k \to i$) begins delivery at time t
y_{bv}	$= 1$ if vessel v is assigned to berth section b
ψ_{bv}	$= 1$ if the leading section of vessel v is placed at section b
$Z^1_{vv'} \dots Z^4_{vv'}$	Sequencing binaries: $= 1$ if vessel v is scheduled before v', for the four ordered combinations of dispatched/received vessels
Q_{ijtv}	$= 1$ if vessel v (route $i \to j$) is berthed at port i at time t

4 Mathematical Model

4.1 Objective Function

$$\min \underbrace{\sum_{m \in N} \sum_{i \in H} C_{mi}\, NTP_m\, x_{mi}}_{\text{first-mile trucking}} + \underbrace{\sum_{n \in N} \sum_{j \in H} C_{nj}\, NTD_n\, x_{nj}}_{\text{last-mile trucking}} + \underbrace{\sum_{i \in H} \sum_{j \in H} \sum_{v \in NV_{ij}} \left[C_{ij} + RV + PP_{ijv} + PD_{ijv} \right]}_{\text{inter-hub vessel cost}} \tag{1}$$

The objective has three parts: the cost of trucking cargo from origin nodes m to their hubs i; the cost of trucking from hubs j to destination nodes n; and the total cost of the inter-hub vessels, comprising transfer, charter, loading, and delivery costs.

4.2 Timing and Synchronization

$$A_{ijv} = tSP_{ijtv} + PP_{ijv} + td_{ijv} \qquad \forall\, i,j \in H,\ v \in NV_{ij},\ t \in T,\ i \neq j \tag{2}$$
$$D_{ijv} = tSD_{ijtv} + PD_{ijv} \qquad \forall\, i,j \in H,\ v \in NV_{ij},\ t \in T,\ i \neq j \tag{3}$$
$$tSD_{ijtv} \geq A_{ijv} \qquad \forall\, i,j \in H,\ v \in NV_{ij},\ t \in T,\ i \neq j \tag{4}$$
$$tt_{mi} + Dt_{mi} \leq A_{iiv} \qquad \forall\, m \in N,\ i \in H,\ v \in NV_{ij} \tag{5}$$
$$tSD_{ijtv} \geq A_{iiv} \qquad \forall\, i,j \in H,\ v \in NV_{ij},\ t \in T \tag{6}$$
$$CD_{mijn} = D_{ijv} + Pt_{nj} + tt_{nj} \qquad \forall\, i,j \in H,\ m,n \in N,\ v \in NV_{ij},\ i \neq j \tag{7}$$
$$CD_{miin} = tt_{mi} + Dt_{mi} + Pt_{ni} + tt_{ni} \qquad \forall\, i \in H,\ m,n \in N \tag{8}$$

$$CD_{miin} \le At_{mn} \qquad \forall\, i \in H,\ m, n \in N \tag{9}$$

Constraint (2) sets the arrival time as the start of loading plus the loading duration plus the sailing time. Constraint (3) sets the delivery-completion time as the delivery start plus its duration. Constraint (4) requires delivery to begin no earlier than arrival. Constraint (5) makes the vessel's availability at its origin at least the largest truck arrival-plus-unloading time. Constraint (6) requires work on the vessel to start no earlier than its availability. Constraint (7) gives the time cargo reaches customer n as the delivery completion at hub j plus truck loading and hub→node travel. Constraint (8) gives the single-hub case directly (truck $m \to i$, unload, load, truck $i \to n$). Constraint (9) enforces the customer order deadline.

4.3 Hub Location, Allocation and Flow

$$\sum_{i \in H} x_{ii} = P \tag{10}$$

$$\sum_{i \in H} x_{mi} = 1 \qquad \forall\, m \in N \tag{11}$$

$$x_{mi} \le x_{ii} \qquad \forall\, m \in N,\ i \in H \tag{12}$$

$$\sum_{i \in H} NV_{ij} \le MNV \qquad \forall\, j \in H,\ i \ne j \tag{13}$$

$$\sum_{\substack{j \in H \\ i \ne j}} f_{ijm} - \sum_{\substack{j \in H \\ i \ne j}} f_{jim} = O_m\, x_{mi} - \sum_{n \in N} W_{mn}\, x_{ni} \qquad \forall\, i \in H,\ m \in N \tag{14}$$

$$\sum_{m \in N} f_{ijm} \le CB \cdot NV_{ij} \qquad \forall\, i, j \in H \tag{15}$$

Constraint (10) opens exactly P hubs. Constraint (11) assigns each node to one hub. Constraint (12) permits assignment to i only if i is a hub. Constraint (13) caps the chartered vessels per inter-hub link at MNV. Constraint (14) is the network flow-balance equation. Constraint (15) ties the number of vessels to the flow they must carry, bounded by vessel capacity CB.

4.4 Berth Allocation and Non-Overlap Scheduling

For every pair of vessels that share a berth section, a disjunctive (big-M) constraint forces one to finish before the other starts. The four relations cover the ordered combinations of dispatched (route $i \to j$) and received (route $k \to i$) vessels.

$$\textstyle\sum_t t\, SP_{ijtv'} \ge M\big(y_{bv} + y_{bv'} + Z^1_{vv'} - 3\big) + \sum_t t\, SP_{ijtv} + PP_{ijv} \qquad \forall\, i, j \in H,\ v, v' \in NV_{ij},\ b \in B_i \tag{16}$$

$$\textstyle\sum_t t\, SD_{kitv'} \ge M\big(y_{bv} + y_{bv'} + Z^2_{vv'} - 3\big) + \sum_t t\, SD_{kitv} + PD_{kiv} \qquad \forall\, k, i \in H,\ v, v' \in NV_{ki},\ b \in B_i \tag{17}$$

$$\textstyle\sum_t t\, SP_{ijtv} \ge M\big(y_{bv} + y_{bv'} + Z^3_{vv'} - 3\big) + \sum_t t\, SD_{kitv'} + PD_{kiv} \qquad \forall\, i, j, k \in H,\ v \in NV_{ij},\ v' \in NV_{ki},\ b \in B_i \tag{18}$$

$$\textstyle\sum_t t\, SP_{kitv} \ge M\big(y_{bv} + y_{bv'} + Z^4_{vv'} - 3\big) + \sum_t t\, SP_{ijtv'} + PP_{ijv} \qquad \forall\, i, j, k \in H,\ v \in NV_{ki},\ v' \in NV_{ij},\ b \in B_i \tag{19}$$

$$\textstyle\sum_{b'=b}^{b+n_{bv}-1} y_{b'} \geq n_{bv} - M(1-\psi_{bv}) \qquad \forall\, b \in B_i,\; v \in NV_{ij},\; i \neq j \tag{20}$$

$$\textstyle\sum_{b'=b}^{b+n_{bv}-1} y_{b'} \leq n_{bv} + M(1-\psi_{bv}) \qquad \forall\, b \in B_i,\; v \in NV_{ij},\; i \neq j \tag{21}$$

$$\sum_{b \in B_i} \psi_{bv} = 1 \qquad \forall\, v \in NV_{ij} \tag{22}$$

$$\sum_{b \in B_i} \psi_{bv} = 1 \qquad \forall\, v \in NV_{ki} \tag{23}$$

$$\textstyle\sum_{b'=b}^{b+n_{bv}-1} y_{b'} \geq n_{bv} - M(1-\psi_{bv}) \qquad \forall\, b \in B_i,\; v \in NV_{ki},\; i \neq k \tag{24}$$

$$\textstyle\sum_{b'=b}^{b+n_{bv}-1} y_{b'} \leq n_{bv} + M(1-\psi_{bv}) \qquad \forall\, b \in B_i,\; v \in NV_{ki},\; i \neq k \tag{25}$$

$$Z^1_{vv'} + Z^1_{v'v} \leq 1 \qquad \forall\, v, v' \in NV_{ij} \tag{26}$$

$$Z^2_{vv'} + Z^2_{v'v} \leq 1 \qquad \forall\, v, v' \in NV_{ki} \tag{27}$$

$$Z^3_{vv'} + Z^3_{v'v} \leq 1 \qquad \forall\, v \in NV_{ij},\; v' \in NV_{ki} \tag{28}$$

$$Z^4_{vv'} + Z^4_{v'v} \leq 1 \qquad \forall\, v \in NV_{ki},\; v' \in NV_{ij} \tag{29}$$

$$\sum_{t \in T} SP_{ijtv} = 1 \qquad \forall\, i, j \in H,\; v \in NV_{ij} \tag{30}$$

$$\sum_{t \in T} SD_{kitv} = 1 \qquad \forall\, i, k \in H,\; v \in NV_{ki} \tag{31}$$

$$\textstyle\sum_t t\, SD_{ijtv} \geq \sum_t t\, SP_{ijtv} + PP_{ijv} + td_{ijv} \qquad \forall\, i, j \in H,\; v \in NV_{ij} \tag{32}$$

$$SP_{ijtv} \geq Q_{ijtv} \qquad \forall\, i, j \in H,\; v \in NV_{ij} \tag{33}$$

$$SD_{kitv} \geq Q_{kitv} \qquad \forall\, i, k \in H,\; v \in NV_{ki} \tag{34}$$

$$\sum_{v \in NV_{ij}} n_{bv} + \sum_{v \in NV_{ki}} n_{bv} \leq L_i \qquad \forall\, i \in H \tag{35}$$

$$\sum_{t \in T} Q_{ijtv} = PP_{ijv} \qquad \forall\, i, j \in H,\; v \in NV_{ij} \tag{36}$$

$$\sum_{t \in T} Q_{kitv} = PD_{kiv} \qquad \forall\, i, k \in H,\; v \in NV_{ki} \tag{37}$$

Constraints (16)–(19) are the berth non-overlap relations: when two vessels share a section ($y_{bv} = y_{bv'} = 1$) and the sequencing binary fixes their order ($Z = 1$), the later vessel may start only after the earlier one finishes; the four versions cover dispatched–dispatched, received–received, received–dispatched, and dispatched–received pairs. Constraints (20), (21) and (24), (25) link the section-occupancy variables y to the leading-section placement ψ, fixing the contiguous block of n_{bv} sections each vessel uses. Constraints (22) and (23) assign exactly one leading section to each dispatched and received vessel. Constraints (26)–(29) enforce sequencing consistency, so at most one vessel of a pair is served first. Constraints (30) and (31) require each vessel to begin loading and delivery at exactly one time. Constraint (32) orders the delivery start after loading start plus processing and sailing. Constraints (33) and (34) require processing to begin no earlier than berthing. Constraint (35) caps the total berth sections used at the quay length L_i. Constraints (36) and (37) equate each vessel's berthing duration to its loading/delivery processing duration.

5 Summary

The model is a mixed-integer program that integrates three decision layers of a maritime logistics network into a single optimization. At the strategic layer it locates P hubs and assigns every node to one of them (single allocation); at the tactical layer it charters and routes capacitated vessels across inter-hub links subject to a flow-balance network design; and at the operational layer it allocates berth sections and schedules loading and delivery in continuous interaction with truck arrivals and customer deadlines. The objective minimizes the combined first-mile trucking, last-mile trucking, and inter-hub vessel cost. The defining technical feature is the tight time synchronization across the truck–vessel–truck chain (constraints 2–9, 32), together with the disjunctive big-M berth-allocation logic (constraints 16–37) that prevents vessels sharing a quay section from being processed simultaneously and ties berthing duration to processing duration.

Scope note. The source material for this project comprises the model formulation only—sets, parameters, decision variables, the objective, and constraints (2)–(37). It does not include a problem-narrative chapter, a literature review, or a solution method and computational results. Accordingly this document reconstructs and documents the formulation faithfully; the assumptions listed in Section 2 are those embedded in the constraints. A small number of source symbols (notably O_m and W_{mn}) were left without written definitions and are interpreted here from their role in the flow-balance equation (14).