A Multi-Dock Cross-Docking Vehicle Routing Problem with Stochastic Travel Times

Mathematical Model and Solution Methodology

1 Overview

This document presents the mathematical formulation and solution methodology of a two-level vehicle routing problem for a multi-dock cross-docking distribution center. The model extends the closest prior work by Chen et al. (2016) in one critical direction: whereas Chen et al. assumed that transfer and travel times inside the cross-docking network are deterministic, this research treats them as **stochastic (probabilistic)**, reflecting real-world uncertainty caused by weather, road conditions, and traffic. Ignoring this uncertainty, as the authors argue, produces routes in which deliveries arrive systematically too early or too late, leading to customer dissatisfaction.

The decision problem is: given a distribution center with several cross-docking terminals, a set of supplier nodes, and a set of customer nodes, determine (i) the routes of pickup vehicles from suppliers to cross-docks, (ii) the routes of delivery vehicles from cross-docks to customers, (iii) the inter-dock product transfers needed when a single dock cannot meet all demand assigned to it, and (iv) the assignment of each vehicle to a specific cross-dock — so that total operational cost is minimized while time windows are respected.

Because the problem is NP-hard and grows exponentially with size, exact solvers (GAMS, branch-and-bound) are impractical for medium and large instances. A **Biogeography-Based Optimization (BBO)** metaheuristic is therefore proposed, with a three-matrix solution encoding designed for the multi-level structure of the problem.

2 Problem Description

2.1 Network Structure

The network contains three types of nodes:

- **Supplier (pickup) nodes** S_p: each holds a known quantity of one product type. Pickup vehicles collect these products and carry them to a cross-dock.
- **Cross-docking terminals** S_0: intermediate hubs at which products are unloaded, sorted, and labeled before being reloaded onto delivery vehicles. There are N_{dock} such terminals.
- **Customer (delivery) nodes** S_d: each requires specified quantities of multiple product types. Delivery vehicles transport products from a cross-dock to customers.

Each pickup vehicle is assigned to exactly one cross-dock and operates a subtour that starts at that dock, visits a subset of supplier nodes, and returns. Each delivery vehicle is similarly assigned to one cross-dock and covers a subset of customer nodes. If the products arriving at a cross-dock from suppliers are insufficient to meet the demand of the customers assigned to it, the shortfall is covered by **inter-dock transfers** from other terminals (at a material-check cost per unit).

2.2 Stochastic Travel Times

The defining novelty of the model is that vehicle travel times on all arcs — both on the supplier-to-dock routes and on the dock-to-customer routes — are treated as random variables with a known probability distribution. As a result, whether a customer's demand is delivered within the required time window cannot be guaranteed with certainty. The model captures this through a stochastic time-window constraint (equations 15 and 16) that bounds the sum of estimated travel times and service times.

2.3 Assumptions

1. Each supplier provides exactly one product type; each customer demands multiple product types.
2. A vehicle (pickup or delivery) may visit a node at most once and is assigned to exactly one cross-dock.
3. Product flow is single-allocation: products collected for a particular dock are routed exclusively through that dock unless an inter-dock transfer is required.
4. Each cross-dock operates under a fixed operational cost C_d per vehicle assigned to it; inter-dock transfers incur a per-unit inspection cost C_h.
5. Travel times are stochastic; deterministic estimates et_{ij} are used in the time-window constraints as expected values.

3 Notation

Symbol	Definition
Sets and indices	
S_p	Set of all pickup (supplier) nodes; index i or j
S_d	Set of all delivery (customer) nodes; index i or j
S_0	Set of all cross-docking terminals; index da, db
N_p	Total number of pickup nodes, $\|S_p\|$
N_d	Total number of delivery nodes, $\|S_d\|$
N_{dock}	Total number of cross-docking terminals
k	Vehicle index
Cost parameters	
ptc_{ij}	Travel cost per pickup vehicle on arc $i \to j$
dtc_{ij}	Travel cost per delivery vehicle on arc $i \to j$
pc_k	Operating cost of pickup vehicle k
dc_k	Operating cost of delivery vehicle k
C_d	Fixed operating cost per vehicle assigned to a cross-dock
C_h	Per-unit material-inspection cost for inter-dock transfers
Capacity and demand parameters	
V_p, V_d	Total number of pickup and delivery vehicles
Q_p, Q_d	Maximum capacity of a pickup and a delivery vehicle

Symbol	Definition
P_i	Quantity of product collected at pickup node i
D_{ij}	Quantity of product type i demanded at delivery node j
Time parameters	
t_i	Service time (loading, unloading, or cross-docking) at node i
et_{ij}	Estimated (expected) travel time from node i to node j
TW_{pick}	Time-window horizon for the pickup operation
TW_{del}	Time-window horizon for the delivery operation
Auxiliary flow variables	
u_{ij}	Units carried by a pickup vehicle on arc $i \to j$
z_{ij}	Units carried by a delivery vehicle on arc $i \to j$
Binary decision variables	
x_{ijk}	$= 1$ if pickup vehicle k travels from node i to node j; else 0
w_{ijk}	$= 1$ if delivery vehicle k travels from node i to node j; else 0
$VP_{da,k}$	$= 1$ if pickup vehicle k is assigned to cross-dock da; else 0
$VD_{da,k}$	$= 1$ if delivery vehicle k is assigned to cross-dock da; else 0
Continuous decision variable	
$q_{da,db,i}$	Quantity of product type i transferred from cross-dock da to cross-dock db; ≥ 0

4 Mathematical Model

4.1 Objective Function

$$\min \underbrace{\sum_{j}\sum_{i}\sum_{k=1}^{V_p} ptc_{ij}\, x_{ijk}}_{\text{pickup transport}} + \underbrace{\sum_{j}\sum_{i}\sum_{k=1}^{V_d} dtc_{ij}\, w_{ijk}}_{\text{delivery transport}} + \underbrace{\sum_{j}\sum_{k=1}^{V_p} pc_k\, x_{0jk}}_{\text{pickup vehicle op.}} + \underbrace{\sum_{j}\sum_{k=1}^{V_d} dc_k\, w_{0jk}}_{\text{delivery vehicle op.}} \tag{1}$$
$$+ \underbrace{C_h \sum_{\substack{b=1\\ b\neq a}}^{N_{\text{dock}}} \sum_{a=1}^{N_{\text{dock}}} \sum_{i} q_{da,db,i}}_{\text{inter-dock inspection}} + \underbrace{C_d \sum_{k=1}^{V_p} \sum_{a=1}^{N_{\text{dock}}} VP_{da,k} \sum_{j=1}^{N_p} x_{0jk} + C_d \sum_{k=1}^{V_d} \sum_{a=1}^{N_{\text{dock}}} VD_{da,k} \sum_{j=1}^{N_d} w_{0jk}}_{\text{cross-dock operating costs}}$$

The objective has six terms: pickup transportation cost; delivery transportation cost; pickup vehicle operating cost (charged per vehicle dispatched from the depot); delivery vehicle operating cost; per-unit material-inspection cost on all inter-dock product transfers; and the fixed operating cost of each cross-dock terminal, weighted by the number of pickup and delivery vehicles assigned to it.

4.2 Routing Constraints

$$\sum_{i}\sum_{k=1}^{V_p} x_{ijk} = 1, \qquad \forall\, j \in S_p \tag{2}$$

$$\sum_{j}\sum_{k=1}^{V_p} x_{ijk} = 1, \qquad \forall\, i \in S_p \tag{3}$$

$$\sum_{i}\sum_{k=1}^{V_d} w_{ijk} = 1, \qquad \forall\, j \in S_d \tag{4}$$

$$\sum_{j}\sum_{k=1}^{V_d} w_{ijk} = 1, \qquad \forall\, i \in S_d \tag{5}$$

$$\sum_{i} x_{i\ell k} - \sum_{j} x_{\ell jk} = 0, \qquad \forall\, \ell,\, k \tag{6}$$

$$\sum_{i} w_{i\ell k} - \sum_{j} w_{\ell jk} = 0, \qquad \forall\, \ell,\, k \tag{7}$$

$$\sum_{k=1}^{V_p}\sum_{i=1}^{N_p} x_{0ik} \le V_p \tag{8}$$

$$\sum_{k=1}^{V_d}\sum_{j=1}^{N_d} w_{0jk} \le V_d \tag{9}$$

Constraints (2)–(5) enforce single-visit requirements: exactly one pickup vehicle enters and exits each supplier node, and exactly one delivery vehicle enters and exits each customer node. Constraints (6) and (7) are vehicle-flow-conservation equations ensuring route continuity (a vehicle entering a node must also leave it). Constraints (8) and (9) cap the number of vehicles dispatched by the available fleet sizes V_p and V_d.

4.3 Capacity Constraints

$$u_{ij} \le Q_p\, x_{ijk}, \qquad \forall\, i, j, k \tag{10}$$

$$z_{ij} \le Q_d\, w_{ijk}, \qquad \forall\, i, j, k \tag{11}$$

Constraints (10) and (11) ensure that units loaded onto a pickup or delivery vehicle on any arc do not exceed the respective vehicle capacities Q_p and Q_d. Because x_{ijk} (or w_{ijk}) is zero when the vehicle does not use that arc, these constraints also enforce that no load is carried on unused arcs.

4.4 Cross-Dock Product-Flow Balance

$$\sum_{k=1}^{V_p}\sum_{j} VP_{da,k}\, P_i\, x_{ijk} + \sum_{\substack{b=1\\ b\neq a}}^{N_{\text{dock}}} q_{db,da,i} - \sum_{\substack{b=1\\ b\neq a}}^{N_{\text{dock}}} q_{da,db,i} = \sum_{k=1}^{V_d}\sum_{j}\sum_{\ell} VD_{da,k}\, D_{ij}\, w_{ijk}, \quad \forall\, i,\, da \tag{12}$$

For each cross-dock da and each product type i, constraint (12) is a product-inventory balance: the products of type i delivered to dock da by its assigned pickup vehicles, plus the products received in inter-dock transfers from other docks, minus the products sent out to other docks in inter-dock transfers, must equal the total demand for product i served by delivery vehicles assigned to dock da.

4.5 Load Propagation Along Routes

$$u_{j\ell} - u_{ij} = \begin{cases} P_j & j \in S_p \\ -\sum_{i=1}^{N_p} P_i & j \in S_0 \end{cases} \tag{13}$$

$$z_{ij} - z_{j\ell} = \begin{cases} \sum_{\text{all } i} D_{ij} & j \in S_d \\ -\sum_{\text{all } i} \sum_{j=1}^{N_d} D_{ij} & j \in S_0 \end{cases} \tag{14}$$

Constraints (13) and (14) track the cumulative product load carried by pickup and delivery vehicles as they traverse their routes. At a supplier node, the pickup load increases by P_j; at a cross-dock, the vehicle delivers all collected products (load decreases by the total). Similarly, at a customer node the delivery load decreases by the total demand served there; at a cross-dock the vehicle loads the full outbound quantity.

4.6 Time-Window Constraints

$$\sum_i \sum_j t_{ik}\, x_{ijk} \;+\; \sum_i \sum_j et_{ijk}\, x_{ijk} \;\le\; TW_{\text{pick}}, \quad \forall\, k \tag{15}$$

$$\sum_i \sum_j t_{ik}\, w_{ijk} \;+\; \sum_i \sum_j et_{ijk}\, w_{ijk} \;\le\; TW_{\text{del}}, \quad \forall\, k \tag{16}$$

For each vehicle k, the total service time (loading, unloading, or cross-docking at visited nodes) plus the total estimated travel time along its route must not exceed the planning-horizon time windows TW_{pick} and TW_{del} for pickup and delivery operations respectively. Because travel times are stochastic, et_{ijk} represents the expected travel time on arc (i, j) for vehicle k; this approximation embeds the stochastic nature of travel into the deterministic constraint structure.

4.7 Vehicle-to-Dock Assignment

$$\sum_{a=1}^{N_{\text{dock}}} VP_{da,k} = 1, \quad \forall\, k \tag{17}$$

$$\sum_{a=1}^{N_{\text{dock}}} VD_{da,k} = 1, \quad \forall\, k \tag{18}$$

Constraints (17) and (18) enforce single assignment: every pickup vehicle and every delivery vehicle is linked to exactly one cross-docking terminal.

4.8 Domain Constraints

$$x_{ijk},\; w_{ijk},\; VP_{da,k},\; VD_{da,k} \in \{0, 1\}; \qquad q_{da,db,i} \ge 0 \tag{19}$$

5 Solution Methodology: Biogeography-Based Optimization

Because the problem is NP-hard and its solution space grows exponentially with the number of nodes, exact methods (e.g. GAMS) become computationally intractable for medium and large instances. The proposed solution approach is the **Biogeography-Based Optimization (BBO)** metaheuristic, introduced by Simon (2008), which models the migration and evolution of species across geographical habitats to drive a population of candidate solutions toward optimality.

5.1 Biogeographic Analogy

Each candidate solution is a *habitat*, described by a vector of suitability index variables (SIVs) that encode the routing decisions:

$$\text{Habitat} = [\text{SIV}_1, \text{SIV}_2, \dots, \text{SIV}_n]. \tag{20}$$

The quality of a habitat is its *Habitat Suitability Index* (HSI):

$$\text{HSI} = f(\text{Habitat}) = f(\text{SIV}_1, \dots, \text{SIV}_n). \tag{21}$$

Habitats with high HSI host many species; they have a high emigration rate (they share good features with weaker habitats) and a low immigration rate (they are near capacity). Optimization proceeds by migrating features from high-HSI habitats into low-HSI ones, interleaved with random mutation.

5.2 Migration and Mutation Rates

For a habitat holding k species out of a maximum of n, the immigration rate λ_k and emigration rate μ_k are

$$\lambda_k = I\left(1 - \frac{k}{n}\right), \qquad \mu_k = E\,\frac{k}{n}, \tag{22}$$

where I and E are the maximum immigration and emigration rates ($E = I$ in practice). The mutation probability for a habitat with S species is

$$m_S = m_{\max}\left(1 - \frac{P_S}{P_{\max}}\right), \tag{23}$$

where P_S is the probability that the habitat contains exactly S species and $P_{\max}$ is its maximum value.

5.3 Three-Matrix Solution Encoding

Because the problem is multi-level, a single solution is encoded as **three matrices** simultaneously:

1. **Supplier-VRP matrix**: encodes the subtour structure of pickup vehicles — each row is a subtour starting with the cross-dock index, followed by the supplier nodes visited in order.
2. **Customer-VRP matrix**: encodes the subtour structure of delivery vehicles in the same way.
3. **Inter-dock transfer matrix**: encodes the quantities $q_{da,db,i}$ of each product type transferred between every pair of cross-docks.

To generate an initial solution, p subtours are selected at random from the n network nodes: for each subtour (row), the center node is placed first, and the remaining nodes are drawn without replacement from the network. Each node may appear in exactly one subtour.

5.4 Fitness Evaluation and Elite Selection

Each habitat is scored using a normalized fitness index. Because the objective is minimization, the index

$$\mu = \frac{1/f_i}{1/f_{\text{best}}} \in [0, 1] \tag{24}$$

assigns higher values to better solutions (lower cost), where f_{best} is the cost of the best solution in the current population. The n habitats with the highest μ are retained as the elite set for the next generation; a further n_2 habitats enter the mutation stage.

5.5 Five Mutation Operators

All five operators work on the subtour representation of one or both VRP matrices. After any move, the fitness of the resulting population is recomputed.

M1. Cross-subtour swap. Select two subtours at random; draw one node from each and swap them between subtours.

M2. Node relocation. Select two subtours; remove one node from the first subtour and insert it into the second.

M3. Single-node split. Select one subtour; detach one of its nodes to form a new singleton subtour.

M4. Two-node split. Select one subtour; detach two of its nodes, which together form a new two-node subtour.

M5. Within-subtour swap. Select one subtour; choose two of its internal nodes and swap their positions within the subtour.

5.6 Algorithm Flow

The complete BBO procedure for this problem proceeds as follows:

1. **Initialize parameters**: maximum species count $S_{\max}$, maximum migration rates $E = I$, maximum mutation rate $m_{\max}$, elitism count.
2. **Generate initial population**: construct random three-matrix habitats by the subtour-sampling procedure described above.
3. **Evaluate HSI** (cost) for every habitat using the objective function (1).
4. **Sort and select elites**: rank habitats by μ; retain the top-n as elites.
5. **Compute migration rates** λ_k and μ_k for each habitat.
6. **Migration**: modify non-elite habitats by importing SIVs from elite habitats, with probabilities proportional to λ and μ.
7. **Mutation**: apply one of the five operators above to n_2 selected habitats with probability m_S.

8. **Repeat** steps 3–7 until the stopping criterion (maximum iterations) is reached.
9. **Return** the best habitat (lowest cost) found.

6 Summary

The model is a nonlinear integer program for a two-level, multi-dock cross-docking vehicle routing problem with stochastic travel times. Its key distinguishing features relative to the deterministic literature (Chen et al., 2016) are: (i) travel times on all arcs are treated as random variables, with time-window compliance expressed through expected values; and (ii) the model explicitly includes inter-dock product transfers, so that demand shortfalls at one terminal can be covered by surplus at another, at an inspection cost. The objective minimizes the sum of pickup and delivery transportation costs, vehicle operating costs, inter-dock material-handling costs, and cross-dock terminal operating costs, over an 18-constraint feasible set that enforces single-visit routing, flow conservation, fleet limits, vehicle capacity, product-flow balance at each cross-dock, load propagation, time windows, and vehicle-to-dock single assignment. Because the problem is NP-hard, a Biogeography-Based Optimization metaheuristic is used as the solution engine, with a three-matrix encoding (supplier-VRP, customer-VRP, inter-dock transfer quantities) and five neighborhood-mutation operators tailored to the subtour structure of the routing component.